On the power structure over the Grothendieck ring of varieties and its applications *

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Abstract

We discuss the notion of a power structure over a ring and the geometric description of the power structure over the Grothendieck ring of complex quasi-projective varieties and show some examples of applications to generating series of classes of configuration spaces (for example, nested Hilbert schemes of J. Cheah) and wreath product orbifolds.

To a pre-λ ring there corresponds a so called *power structure*. This means, in particular, that one can give sense to an expression of the form

$$(1 + a_1 t + a_2 + ...)^m$$

for a_i and m from the ring R. (Generally speaking, on a ring there are many pre-λ structures which correspond to one and the same power structure.) A natural pre-λ structure on the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties is defined by the Kapranov zeta-function

$$\zeta_X(t) = 1 + [X]t + [S^2X]t2 + [S^3X]t3 + \dots ,$$

where $S^kX = X^k/S_k$ is the k-th symmetric power of the variety X. In [8], there was given a geometric description of the corresponding power structure

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over the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$. In some cases this permits to give new (short and somewhat more transparent) proofs and also certain refinements of formulae for generating series of classes of moduli spaces in the ring $K_0(\mathcal{V}_{\mathbb{C}})$ and/or of their invariants: the Euler characteristic and the Hodge–Deligne polynomial. An application of this sort (for the generating series of classes of Hilbert schemes of 0-dimensional subschemes of a smooth quasi-projective variety) was described in [9].

The aim of this paper is to describe the concept of a power structure (in a somewhat more general context introduced in [10]) and to show its applications to proofs and also some improvements of results by J. Cheah in [4] about nested Hilbert schemes, by W.P. Lin and Zh. Qin in [12] about moduli spaces of 1-dimensional subschemes. Finally we rewrite some results of W. Wang, J. Zhou in [15] and [16] on generating series of orbifold generalized Euler characteristicic of wreath product orbifolds in terms of the power structure.

1 Power structures

Definition: A pre-λ structure on a ring R is given by a series $\lambda_a(t) \in 1 + t \cdot R[[t]]$ defined for each $a \in R$ so that

1. $\lambda_a(t) = 1 + at \mod t^2$.
2. $\lambda_{a+b}(t) = \lambda_a(t)\lambda_b(t)$ for $a, b \in R$.

Example. One has the following important examples of pre-λ structures.

1. R is the ring $\mathbb{Z}$ of integers, $\lambda_k(t) = (1-t)^{-k}$.
2. $R = \mathbb{Z}$ and $\lambda_k(t) = (1+t)^k$.
3. $R = \mathbb{Z}[u_1, \ldots, u_r]$ (the ring of polynomials in r variables $u_1, \ldots, u_r$), for a polynomial $P = P(\underline{u}) = \sum p_{\underline{k}}\, \underline{u}^{\underline{k}}$, $\underline{k} \in \mathbb{Z}^r_{\geq 0}$ and $p_{\underline{k}} \in \mathbb{Z}$,
$$\lambda_P(t) = \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} (1 - \underline{u}^{\underline{k}} t)^{-p_{\underline{k}}}\,,$$
where $\underline{u} = (u_1, \ldots, u_r)$, $\underline{k} = (k_1, \ldots, k_r)$, $\underline{u}^{\underline{k}} = u_1^{k_1} \cdot \ldots \cdot u_r^{k_r}$, (see [9]).

4. (A more geometric example.) Let R be the K-functor $K(X)$ of the space X, i.e. the Grothendieck ring of (say, real or complex) vector bundles over X. For a vector bundle E over X, let $\Lambda^k E$ be the k-th exterior power of the bundle E. The series

$$\lambda_E(t) = 1 + [E]t + [\Lambda^2 E]t^2 + [\Lambda^3 E]t^3 + \ldots$$

defines a pre-λ structure on the ring $K(X)$.

To a pre-λ structure on a ring R one can associate a *power structure* over R: a notion introduced in [8].

Definition: A *power structure* over a (semi)ring R with a unit is a map $(1 + t \cdot R[[t]]) \times R \to 1 + t \cdot R[[t]]$: $(A(t), m) \mapsto (A(t))^m$, which possesses the following properties:

1) $(A(t))^0 = 1$,

2) $(A(t))^1 = A(t)$,

3) $(A(t) \cdot B(t))^m = (A(t))^m \cdot (B(t))^m$,

4) $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$,

5) $(A(t))^{mn} = ((A(t))^n)^m$,

6) $(1+t)^m = 1 + mt +$ terms of higher degree,

7) $\left(A(t^k)\right)^m = (A(t))^m \,|_{t \mapsto t^k}$.

Remark. For a ring property 1) follows from the other ones. It is necessary to keep it only for a semiring.

Definition: A power structure is *finitely determined* if for each $M > 0$ there exists a $N > 0$ such that for any series $A(t)$ the M-jet of the series $(A(t))^m$ (i.e., $(A(t))^m \bmod t^{M+1}$) is determined by the N-jet of the series $A(t)$.

Proposition 1 *To define a finitely determined power structure over a ring R it is sufficient to define the series $(A_0(t))^m$ for any fixed series $A_0(t)$ of the form $1 + t +$ terms of higher degree, and for each $m \in R$, so that:*

1) $(A_0(t))^m = 1 + mt +$ terms of higher degree;

2) $(A_0(t))^{m+n} = (A_0(t))^m (A_0(t))^n$.

Proof. By properties 6 and 7, each series $A(t) \in 1 + t \cdot R[[t]]$ can be written in a unique way as a product of the form $\prod_{i=1}^{\infty}(A_0(t^i))^{b_i}$, with $b_i \in R$. Then by properties 3 and 7 (and the finite determinacy of the power structure)

$$(A(t))^m = \prod_{i=1}^{\infty}(A_0(t^i))^{b_i m}. \tag{1}$$

In the other direction, one can easily see that the power structure defined by the equation (1) possesses properties 1)–7). $\square$

Proposition 1 means that a pre-λ structure on the ring R defines a finitely determined power structure over R. In the other direction, there are many pre-λ structures on the ring R which give one and the same power structure: those defined by the series $(A_0(t))^m$ for any fixed series $A_0(t)$ of the form $1 + t + \ldots$ terms of higher degree. In what follows we shall prefer to use the series $A_0(t) = (1-t)^{-1} = 1 + t + t^2 + \ldots \in R[[t]]$.

Let $R[[\underline{t}]] = R[[t_1, \ldots, t_r]]$ be the ring of series in r variables $t_1, \ldots, t_r$ with coefficients from the ring R and let $\mathfrak{m}$ be the ideal $\langle t_1, \ldots, t_r\rangle$. A power structure over the ring R in a natural way permits to give sense to expressions of the form $(A(\underline{t}))^m$, where $A(\underline{t}) \in 1 + \mathfrak{m}R[[\underline{t}]]$. Namely, the series $A(\underline{t})$ can be in a unique way represented in the form

$$A(\underline{t}) = \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}} (1 - \underline{t}^{\underline{k}})^{-b_{\underline{k}}}$$

($\underline{t}^{\underline{k}} = t_1^{k_1} \ldots t_r^{k_r}$). Then

$$(A(\underline{t}))^m = \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}} (1 - \underline{t}^{\underline{k}})^{-b_{\underline{k}} m}\,.$$

Let R_1 and R_2 be rings with power structures over them. A ring homomorphism $\varphi : R_1 \to R_2$ induces the natural homomorphism $R_1[[\underline{t}]] \to R_2[[\underline{t}]]$ (also denoted φ) by $\varphi\left(\sum a_{\underline{i}}\, \underline{t}^{\underline{i}}\right) = \sum \varphi(a_{\underline{i}})\, \underline{t}^{\underline{i}}$. One has:

Proposition 2 *If a ring homomorphism $\varphi : R_1 \to R_2$ is such that $(1-t)^{-\varphi(m)} = \varphi\left((1-t)^{-m}\right)$ for any $m \in R$, then $\varphi\left((A(\underline{t}))^m\right) = \left(\varphi\left(A(\underline{t})\right)\right)^{\varphi(m)}$ for $A(\underline{t}) \in 1 + \mathfrak{m}R[[\underline{t}]]$, $m \in R$.*

Definition: The *Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties* is the abelian group generated by classes $[X]$ of all quasi-projective varieties X modulo the relations:

1) if varieties X and Y are isomorphic, then $[X] = [Y]$;

2) if Y is a Zariski closed subvariety of X, then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\mathcal{V}_{\mathbb{C}})$ is defined by the Cartesian product of varieties.

Remark. One can also consider the concept of the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties substituting the word "group" above by the word "semigroup". Elements of the semiring $S_0(\mathcal{V}_{\mathbb{C}})$ have somewhat more geometric sense: they are represented by "genuine" quasi-projective varieties (not by virtual ones).

The class $[\mathbb{A}^1_{\mathbb{C}}] \in K_0(\mathcal{V}_{\mathbb{C}})$ of the complex affine line is denoted by $\mathbb{L}$. In a number of cases it is reasonable (or rather necessary) to consider the localization $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ of the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ by the class $\mathbb{L}$.

For a complex quasi-projective variety X, let $S^kX = X^k/S_k$ be the k-th symmetric power of the space X (here S_k is the group of permutations on n elements; S^kX is a quasi-projective variety as well).

Definition: The *Kapranov zeta function* of a quasi-projective variety X is the series

$$\zeta_X(t) = 1 + [X] \cdot t + [S^2X] \cdot t^2 + [S^3X] \cdot t^3 + \ldots \in K_0(\mathcal{V}_{\mathbb{C}})[[t]]$$

([11]).

One can see that

$$\zeta_{X+Y}(t) = \zeta_X(t) \cdot \zeta_Y(t)\,. \tag{2}$$

This follows from the relation $S^k(X \amalg Y) = \coprod_{i=0}^{k} S^iX \times S^{k-i}Y$. Also one has

$$\zeta_{\mathbb{L}^n}(t) = \frac{1}{1 - \mathbb{L}^n t}\,.$$

As an example this implies that

$$\zeta_{\mathbb{CP}^n}(t) = \prod_{i=0}^{n} \frac{1}{1 - \mathbb{L}^i t}\,.$$

Equation (2) means that the series $\zeta_X(t)$ defines a pre-λ structure on the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$. The geometric description of the corresponding power structures over the ring $K_0(\mathcal{V}_{\mathbb{C}})$ was given in [8]. We shall formulate it here in the form adapted for series in r variables ([10]).

Let $A_{\underline{n}}$, $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}$, and M be quasi-projective varieties and $A(\underline{t}) = 1 + \sum_{\underline{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} [A_{\underline{n}}]\, \underline{t}^{\underline{n}} \in K_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$. Let $\mathfrak{A}$ be the disjoint union $\coprod_{\underline{k} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} A_{\underline{k}}$, and let $\underline{k} : \mathfrak{A} \to \mathbb{Z}^r_{\geq 0}$ be the tautological map on it: it sends points of $A_{\underline{k}}$ to $\underline{k} \in \mathbb{Z}^r_{\geq 0}$.

Geometric description of the power structure over the ring $K_0(\mathcal{V}_{\mathbb{C}})$. The coefficient at $\underline{t}^{\underline{n}}$ in the series $A(\underline{t})^{[M]}$ is represented by the configuration space of pairs (K, φ), where K is a finite subset of the variety M and φ is a map from K to $\mathfrak{A}$ such that $\sum_{x \in K} \underline{k}(\varphi(x)) = \underline{n}$. To describe such a configuration space as a quasi-projective variety one can write it as

$$\sum_{\mathbf{k}: \sum \underline{i}\, k_{\underline{i}} = \underline{n}} \left[\left((\prod_{\underline{i}} M^{k_{\underline{i}}}) \setminus \Delta \right) \times \prod_{\underline{i}} A_{\underline{i}}^{k_{\underline{i}}} / \prod_{\underline{i}} S_{k_{\underline{i}}} \right], \tag{3}$$

where $\mathbf{k} = \{k_{\underline{i}} : \underline{i} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}, k_{\underline{i}} \in \mathbb{Z}\}$ and Δ is the "large diagonal" in $M^{\Sigma k_{\underline{i}}}$ which consists of $(\sum k_{\underline{i}})$-tuples of points of M with at least two coinciding ones; the permutation group $S_{k_{\underline{i}}}$ acts by permuting corresponding $k_{\underline{i}}$ factors in $\prod_s M^{k_{\underline{i}}} \supset (\prod_{\underline{i}} M^{k_{\underline{i}}}) \setminus \Delta$ and the spaces $A_{\underline{i}}$ simultaneously (the connection between this formula and the description above is clear).

One can shows (see [8]) that the described operation really gives a power structure over $K_0(\mathcal{V}_{\mathbb{C}})$, i.e. it satisfies conditions 1) – 7) of the definition. The fact that this structure corresponds to the Kapranov zeta function follows from the equation

$$(1 + t + t^2 + \ldots)^{[M]} = 1 + [M] \cdot t + [S^2 M] \cdot t^2 + [S^3 M] \cdot t^3 + \ldots \tag{4}$$

Indeed, since there is only one map from M to a point (the coefficients in the series $1 + t + t^2 + \ldots$), the coefficient at t^n in the LHS of equation (4) is represented by the space a point of which is a finite set of points of the variety M with positive multiplicities such that the sum of these multiplicities is equal to n. This is just the definition of the n-th symmetric power of the variety M.

It is also useful to describe the binomial $(1+t)^{[M]}$. The coefficient at t^n in it is represented by the space a point of which is a finite subset of M with n elements, i.e. the configuration space $(M^n \setminus \Delta)/S_n$ of unordered n-tuples of distinct points of M.

It seems that the power structure can be used to prove some combinatorial identities. For instance, applying formula (3) to a finite set M with m elements one gets a formula for the power of a series:

$$\left(1 + \sum_{\underline{n}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}} a_{\underline{n}}\, \underline{t}^{\underline{n}}\right)^m = 1 + \sum_{\underline{n}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}} \left(\sum_{\mathbf{k}:\, \sum \underline{i}k_{\underline{i}}=\underline{n}} \frac{m!}{(m-\Sigma k_{\underline{i}})! \prod_{\underline{i}} k_{\underline{i}}!} \prod_{\underline{i}} a_{\underline{i}}^{k_{\underline{i}}} \right) \underline{t}^{\underline{n}}.$$

There are two natural homomorphism from the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u,v]$ of polynomials in two variables: the Euler characteristic (with compact support) $\chi : K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}$ and the Hodge–Deligne polynomial $e : K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[u,v]$: $e(X)(u,v) = \sum e^{p,q}(X) u^p v^q$.

The formula of I.G. Macdonald [13]:

$$\chi(1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots) = (1-t)^{\chi(X)}$$

and the corresponding formula for the Hodge–Deligne polynomial (see [2], [3, Proposition 1.2]):

$$e(1 + [X]t + [S^2X]t^2 + \ldots)(u,v) = \prod_{p,q} \left(\frac{1}{1-u^pv^qt}\right)^{e^{p,q}(X)}$$

implies that these homomorphisms respect the power structures on these rings described above (see Example 3 and Proposition 2 or [9]). Therefore a relation between series from $K_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$ written in terms of the power structure yields the corresponding relations between the Euler characteristics and the Hodge–Deligne polynomials of these series.

Remark. It is also possible to define the power structure and to describe it in the *relative setting*, i.e. over the Grothendieck ring $K_0(\mathcal{V}_{\mathcal{S}})$ of complex quasi-projective varieties over a variety S. The ring $K_0(\mathcal{V}_{\mathcal{S}})$ is generated by classes of varieties with maps ("projections") to S. In this case the coefficient of the series $(A(\underline{t}))^{[M]}$ is the configuration space a point of which is a pair (K, φ) consisting of a finite subset $K \subset M$ which is contained in the preimage of one point of S and the map φ commutes with the projections to S.

2 Nested Hilbert schemes of J. Cheah

Let Hilb^n_X, $n \geq 1$, be the Hilbert scheme of zero-dimensional subschemes of length n of a complex quasi-projective variety X; for $x \in X$, let $\mathrm{Hilb}^n_{X,x}$ be the Hilbert scheme of subschemes of X supported at the point x.

In [4], J. Cheah considered nested Hilbert schemes on a smooth d-dimensional complex quasi-projective variety X. For $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0}$, the *nested Hilbert scheme* $Z^{\underline{n}}_X$ of depth r is the scheme which parametrizes collections of the form $(Z_1, \ldots, Z_r)$, where $Z_i \in \mathrm{Hilb}^{n_i}_X$ and Z_i is a subscheme of Z_j for $i < j$. The scheme $Z^{\underline{n}}_X$ is non-empty only if $n_1 \leq n_2 \leq \ldots \leq n_r$; notice that $Z^{(n)}_X = \mathrm{Hilb}^n_X \cong Z^{(n,\ldots,n)}_X$.

For $Y \subset X$, let $Z^{\underline{n}}_{X,Y}$ be the scheme which parametrizes collections $(Z_1, \ldots, Z_r)$ from $Z^{\underline{n}}_X$ with $\mathrm{supp}\, Z_i \subset Y$. For $Y = \{x\}$, $x \in X$, we shall use the notation $Z^{\underline{n}}_{X,x}$.

For $r \geq 1$, let $\underline{t} = (t_1, \ldots, t_r)$ and

$$\mathcal{Z}^{(r)}_X(\underline{t}) := \sum_{\underline{n} \in Z^r_{\geq 0}} [Z^{\underline{n}}_X]\, \underline{t}^{\underline{n}}, \quad \mathcal{Z}^{(r)}_{X,x}(\underline{t}) := \sum_{\underline{n} \in Z^r_{\geq 0}} [Z^{\underline{n}}_{X,x}]\, \underline{t}^{\underline{n}},$$

be the generating series of classes of nested Hilbert schemes $Z^{\underline{n}}_X$ (resp. supported at the point x) of depth r.

Theorem 1 *For a smooth quasi-projective variety X of dimension d, the following identity holds in $S_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$ (and therefore also in $K_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$):*

$$\mathcal{Z}^{(r)}_X(\underline{t}) = \left(\mathcal{Z}^{(r)}_{\mathbb{A}^d,0}(\underline{t})\right)^{[X]}. \tag{5}$$

Proof. For a Zariski closed subset $Y \subset X$, one has $\mathcal{Z}^{(r)}_X(\underline{t}) = \mathcal{Z}^{(r)}_{X,Y}(\underline{t}) \cdot \mathcal{Z}^{(r)}_{X,X\setminus Y}(\underline{t})$. Therefore it is sufficient to prove equation (5) for Zariski open subsets U of X which form a covering of X and for their intersections.

One can take U which lies in an affine chart $\mathbb{A}^N_{\mathbb{C}}$ and such that its projection to a d-dimensional coordinate space (say, generated by the first d coordinates) is everywhere non-degenerate (i.e. is an ètale morphism). For any point $x \in U$, this projection identifies $\underline{n}$-nested Hilbert schemes of $Z^{\underline{n}}_{U,x}$ with $Z^{\underline{n}}_{\mathbb{A}^d_{\mathbb{C}},x}$.

A nested (zero-dimensional) subscheme of U of type $\underline{n}$ is defined by a finite subset $K \subset U$ with a nested subscheme from $Z^{\underline{k}(x)}_{X,x}$ at each point $x \in K$.

such that $\sum\limits_{x\in K} \underline{k}(x) = \underline{n}$. This coincides with the description of the coefficient at $\underline{t}^{\underline{n}}$ in the RHS of the equation (5). $\square$

Similar considerations permit to give a short proof of a somewhat refined version of the main result of [4]. Following J. Cheah, let

$$\begin{aligned}
\mathfrak{F}_X^n &= \{(x, Z) \in X \times \mathrm{Hilb}_X^n : \; x \in \operatorname{supp} Z\},\\
\mathfrak{F}_X^{n-1,n} &= \{(x_1, x_2, Z_1, Z_2) \in X \times X \times Z_X^{(n-1,n)} : \; x_i \in \operatorname{supp} Z_i, \; i = 1, 2\},\\
\mathfrak{T}_X^n &= \{(x_1, x_2, Z) \in X \times X \times \mathrm{Hilb}_X^n : \; x_i \in \operatorname{supp} Z, \; i = 1, 2\},\\
\mathfrak{G}_X^n &= \{(x, Z_1, Z_2) \in X \times Z_X^{(n-1,n)} : \; x \in \operatorname{supp} Z_2\}.
\end{aligned}$$

Let the series $\mathfrak{P}_X(t_0, t_1, t_2, t_3)$ and $\mathfrak{f}_d(t_0, t_1, t_2, t_3)$ from $K_0(\mathcal{V}_{\mathbb{C}})[[t_0, t_1, t_2, t_3]]$ be defined by

$$\begin{aligned}
\mathfrak{P}_X(t_0, t_1, t_2, t_3) \;:=\;& \left[\sum_{n\geq 0}[\mathrm{Hilb}_X^n]t_0^n\right] + \left[\sum_{n\geq 1}[\mathfrak{F}_X^n]t_0^n\right] t_1\\
+\;& \left[\sum_{n\geq 1}[\mathfrak{F}_X^n]t_0^n\right] t_2 + \left[\sum_{n\geq 1}[\mathfrak{T}_X^n]t_0^n\right] t_1 t_2\\
+\;& \left[\sum_{n\geq 1}[Z_X^{(n-1,n)}]t_0^n\right] t_3 + \left[\sum_{n\geq 2}[Z_X^{(1,n-1,n)}]t_0^n\right] t_1 t_3\\
+\;& \left[\sum_{n\geq 1}[\mathfrak{G}_X^n]t_0^n\right] t_2 t_3 + \left[\sum_{n\geq 2}[\mathfrak{F}_X^{n-1,n}]t_0^n\right] t_1 t_2 t_3\,,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{f}_d(t_0, t_1, t_2, t_3) \;:=\;& \sum_{k\geq 0}[\mathrm{Hilb}^k_{\mathbb{A}^d,0}]t_0^k + \sum_{k\geq 1}[Z^{(k-1,k)}_{\mathbb{A}^d,0}]t_0^k t_3\\
+\;& \sum_{k\geq 1}[\mathrm{Hilb}^k_{\mathbb{A}^d,0}]t_0^k t_2 + \sum_{k\geq 1}[Z^{(k-1,k)}_{\mathbb{A}^d,0}]t_0^k t_2 t_3\\
+\;& \sum_{k\geq 1}[\mathrm{Hilb}^k_{\mathbb{A}^d,0}]t_0^k t_1 + \sum_{k\geq 1}[\mathrm{Hilb}^k_{\mathbb{A}^d,0}]t_0^k t_1 t_2\\
+\;& \sum_{k\geq 2}[Z^{(k-1,k)}_{\mathbb{A}^d,0}]t_0^k t_1 t_3 + \sum_{k\geq 2}[Z^{(k-1,k)}_{\mathbb{A}^d,0}]t_0^k t_1 t_2 t_3\,.
\end{aligned}$$

Theorem 2 *(cf. Main Theorem in [4]) Let X be a smooth quasi-projective variety of dimension d. Then*

$$\mathfrak{P}_X(t_0, t_1, t_2, t_3) = (\mathfrak{f}_d(t_0, t_1, t_2, t_3))^{[X]} \qquad \mod (t_1^2, t_2^2, t_3^2)\,. \tag{6}$$

Proof. Using the arguments of the proof of Theorem 1 we may suppose that X lies in an affine chart $\mathbb{A}^N_{\mathbb{C}}$ and its projection to a d-dimensional coordinate space is nondegenerate. This identifies $\mathrm{Hilb}^s_{X,x}$ and $Z^{(s-1,s)}_{X,x}$ with $\mathrm{Hilb}^s_{\mathbb{A}^d,0}$ and $Z^{(s-1,s)}_{\mathbb{A}^d,0}$ respectively for each point $x \in X$. To prove equation (6) one has to give an interpretation of the coefficients at the monomials $t_0^n, t_0^n t_1, \ldots, t_0^n t_1 t_2 t_3$ in the RHS of (6). Let us make this for the coefficients at $t_0^n t_3$ and at $t_0^n t_2 t_3$ (other cases are treated in the same way).

The coefficient at $t_0^n t_3$ is represented by the space a point of which is defined by a point x_0 of X with a zero-dimensional nested scheme from $Z^{(k(x_0)-1,k(x_0))}_{X,x_0}$ at it plus several other points of X with zero-dimensional schemes from $\mathrm{Hilb}^{k(x)}_{X,x} \cong Z^{(k(x),k(x))}_{X,x}$ at each of them, such that $k(x_0) + \sum k(x) = n$. This is just the definition of a point of the space $Z^{(n-1,n)}_X$.

The monomial $t_0^n t_2 t_3$ can be obtained either as a product of two monomials of the form $t_0^* t_2$, $t_0^* t_3$ and of several monomials of the form t_0^* or as a product of a monomial of the form $t_0^* t_2 t_3$ and of several monomials of the form t_0^*. Therefore the coefficient at the monomial $t_0^n t_2 t_3$ is represented by the space consisting of two parts.

A point of the first part is defined by a point x_1 of X with a scheme from $\mathrm{Hilb}^{k(x_1)}_{X,x_1} \cong Z^{(k(x_1),k(x_1))}_{X,x_1}$ at it, with $k \geq 1$ (i.e. it is not empty: x_1 belongs to the support of it), a point $x_2 \in X$ with a scheme from $Z^{(k(x_2)-1,k(x_2))}_{X,x_2}$ at it plus several points of X with 0-dimensional schemes from $\mathrm{Hilb}^{k(x)}_{X,x}$ at each of them such that $k(x_1) + k(x_2) + \sum k(x) = n$.

A point of the second part is defined by a point x_1 of X with the scheme (z_1, z_2) from $Z^{(k(x_1)-1,k(x_1))}_{X,x_1}$ at it (in this case z_2 is not empty: x_1 belongs to the support of it) plus several points of X with 0-dimensional schemes from $\mathrm{Hilb}^{k(x)}_{X,x}$ at each of them such that $k(x_1) + \sum k(x) = n$. Therefore a point of the union of these two subspaces can be described by a nested scheme (Z_1, Z_2) from $Z^{(n-1,n)}_X$ plus a point which belongs to Z_2. This is just the description of the space $\mathfrak{G}^n_X$. $\square$

Applying the Hodge–Deligne homomorphism to (6) one gets the Main Theorem of [4].

Example. Let S be a smooth quasi-projective surface. Consider the incidence variety $Z^{(n-1,n)}_S = \{(Z_1, Z_2) \in \mathrm{Hilb}^{n-1}_S \times \mathrm{Hilb}^n_S : Z_1 \subset Z_2\}$. Using the results of J. Cheah on the cellular decomposition of $Z^{(n-1,n)}_{\mathbb{A}^2_{\mathbb{C}},0}$ ([5]), one gets

the result of L. Göttsche ([7, Theorem 5.1]):

$$\sum_{n\geq 1}[Z_S^{n-1,n}]\,t^n = \frac{[S]\cdot t}{1-\mathbb{L}t}\left(\prod_{k\geq 1}\frac{1}{1-\mathbb{L}^{k-1}t^k}\right)^{[S]}.$$

3 On moduli spaces of curves and points - (after W.-P. Li and Zh. Qin)

In [12], there were considered certain moduli spaces of 1-dimensional subschemes in a smooth d-dimensional projective complex variety. Let X be a smooth d-dimensional projective complex variety with a Zariski locally trivial fibration $\mu : X \to S$ where S is smooth of dimension $d-1$ and fibres are smooth irreducible curves of genus g. Let $\beta \in H_2(X,\mathbb{Z})$ be the class of the fibre.

Let $\mathfrak{I}_n(X,\beta)$ be the moduli space of 1-dimensional closed subschemes Z of X such that $\chi(\mathcal{O}_Z)=n, [Z]=\beta$, where $[Z]$ is the fundamental class of the scheme Z and let $\mathfrak{M}^n := \mathfrak{I}_{(1-g)+n}(X,\beta)$. Let $\mathfrak{M}^n_{X,C_s,x}$ be the moduli space of 1-dimensional closed subschemes Θ in X such that $I_\Theta \subset I_{C_s}$, the support $\operatorname{supp}(I_{C_s}/I_\Theta)=\{O\}$ and $\dim_O(I_{C_s}/I_\Theta)=n$. The number n will be called the length of the subscheme Θ. Let $\mathfrak{M}^n_{\mathbb{C}^{d-1}\times\mathbb{C},\{O\}\times\mathbb{C},O}$ have the same meaning: it is the moduli space of 1-dimensional closed subschemes Θ in $\mathbb{C}^{d-1}\times\mathbb{C}$ such that $I_\Theta\subset I_{\{O\}\times\mathbb{C}}$, $\operatorname{supp}\left(I_{\{O\}\times\mathbb{C}}/I_\Theta\right)=\{O\}$ and $\dim_O\left(I_{\{O\}\times\mathbb{C}}/I_\Theta\right)=n$.

Theorem 3 *(cf. Proposition 5.3, Lemma 6.1 and Proposition 6.2 in [12]) Let X be a smooth d-dimensional projective complex variety with a Zariski locally trivial fibration $\mu : X\to S$ where S is smooth of dimension $d-1$ and fibres are smooth irreducible curves of genus g. Then*

$$\sum_{n\geq 0}[\mathfrak{M}^n]\,t^n = [S]\left(\sum_{n\geq 0}[\mathrm{Hilb}^n_{\mathbb{C}^d,0}]\,t^n\right)^{[X]-[C]}\left(\sum_{n\geq 0}[\mathfrak{M}^n_{\mathbb{C}^{d-1}\times\mathbb{C},\{O\}\times\mathbb{C},O}]t^n\right)^{[C]}. \tag{7}$$

Proof. A point of $\mathfrak{M}^n$ can be considered as consisting of a fibre $C_s=\mu^{-1}(s)$ of the bundle $\mu : X\to S$ and of several fixed points, both outside of C_s and on it, with a 0-dimensional subscheme (i.e. an element of $\mathrm{Hilb}^*_{X,x}$) at each of

those points which are outside of C_s and a subscheme of $\mathfrak{M}^*_{X,C_s,x}$ at each of those points which lie on C_s such that the sum of their lengths is equal to n. Thus, there is a natural map (projection) from $\mathfrak{M}^n$ to S. Over a point $s \in S$, there are somewhat different objects (subschemes) at points outside of the curve C_s and on this curve.

It is sufficient to prove equation (7) for preimages of elements of a covering of S by Zariski open subsets and of their intersections. Therefore without any loss of generality we can suppose that $X = S \times C$. Moreover, let us choose a fixed point $s_0 \in S$. A constructible map which sends $\mathfrak{M}^n_{X,C_s}$ to $\mathfrak{M}^n_{X,C_{s_0}}$ and is an isomorphism of strata can be defined as follows. One takes a 0-dimensional subscheme which lies on C_{s_0} and puts them to the corresponding points of C_s and vice versa, one takes the elements of $\mathfrak{M}^n_{X,C_s,x}$ and puts them to the corresponding points of C_{s_0}. Thus in the Grothendieck ring of algebraic varieties one has $[\mathfrak{M}^n] = [S][\mathfrak{M}^n_{X,C_{s_0}}]$.

Therefore to prove (7) one should show that

$$\sum_{n\geq 0}[\mathfrak{M}^n_{X,C_{s_0}}]\, t^n = \left(\sum_{n\geq 0}[\mathrm{Hilb}^n_{\mathbb{C}^d,0}]\, t^n\right)^{[X]-[C_{s_0}]} \left(\sum_{n\geq 0}[\mathfrak{M}^n_{\mathbb{C}^{d-1}\times\mathbb{C},\{O\}\times\mathbb{C},O}]\, t^n\right)^{[C_{s_0}]}. \tag{8}$$

Just as in the proofs in Section 2 we may suppose that at each point of X the space $\mathrm{Hilb}^k_{X,x}$ is identified with the space $\mathrm{Hilb}^k_{\mathbb{C}^d,0}$ and at each point of $C_{s_0} \subset X$ the space $\mathfrak{M}^k_{X,C_{s_0},x}$ is identified with the space $\mathfrak{M}^k_{\mathbb{C}^{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},0}$. The coefficient at t^n in the RHS of equation (8) is represented by the space a point of which is defined by several points of the curve $C_{s_0} \subset X$ with a scheme from $\mathfrak{M}^{k(x)}_{X,C_{s_0},x}$ at each of them and several points from $X \setminus C_{s_0}$ with a scheme from $\mathrm{Hilb}^{k(x)}_{X,x}$ at each of them such that the sum of the lengths $k(x)$ over all the mentioned points is equal to n. This is just the description of a point of $\mathfrak{M}^n_{X,C_{s_0}}$. $\square$

4 Orbifold generalized Euler characteristic and the power structure

Here we rewrite some results of [15] and [16] in terms of the power structure. For that we need it over a somewhat modified version of the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$. For a fixed positive integer m, consider the ring $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{1/m}]$.

The pre-λ structure on (and therefore the corresponding power structure over) the ring $K_0(\mathcal{V}_{\mathbb{C}})$ can be extended to one on $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ by the formula

$$\zeta_{[X]\mathbb{L}^{s/m}}(t) = \zeta_{[X]}(\mathbb{L}^{s/m}t)\,.$$

In a similar way the corresponding pre-λ structure on the ring $\mathbb{Z}[u_1^{1/m}, \ldots, u_r^{1/m}]$ can be defined by the formula

$$\lambda_P(t) = \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} (1-\underline{u}^{\underline{k}}t)^{-p_{\underline{k}}}$$

for a polynomial $P = P(\underline{u}) = \sum\limits_{\underline{k}\in(1/m)\mathbb{Z}^r_{\geq 0}} p_{\underline{k}}\, \underline{u}^{\underline{k}}$. There are natural homomorphisms (χ and e) from the ring $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ to the rings $\mathbb{Z}$ and $\mathbb{Z}[u^{1/m}, v^{1/m}]$ which send the element $\mathbb{L}^{1/m}$ to 1 and $(uv)^{(1/m)}$ respectively. One can easy see that these are homomorphisms of the λ-rings and therefore they respect the power structures.

Let X be a smooth quasi-projective complex algebraic variety of dimension d with an action of a finite group G of order m. For an element $g \in G$, let X^g be the set $\{x \in X : gx = x\}$ of g-invariant points of the action. If $h = vgv^{-1}$ in G, the element v defines an isomorphism $v : X^g \to X^h$. Let G_* be the set of conjugacy classes of elements of the group G. For a conjugacy class $c \in G_*$ choose its representative $g \in G$. Let $C_G(g)$ be the centralizer of the element g in G. The centralizer $C_G(g)$ acts on the set X^g of fixed points of g. Suppose that its action on the set of connected components of X^g has N_c orbits and let $X_1^g, \ldots, X_{N_c}^g$ be unions of components of each of these orbits. At each point $x \in X_{\alpha_c}^g$, the map dg is an automorphism of the tangent space T_xX which acts as a diagonal matrix $\mathrm{diag}\,(\exp(2\pi i\theta_1), \ldots, \exp(2\pi i\theta_d))$, where $0 \leq \theta_i < 1$, $\theta!_i \in (1/m)\mathbb{Z}$. The *shift number* $F_{\alpha_c}^g$ associated to $X_{\alpha_c}^g$ is $F_{\alpha_c}^g := \sum_{j=1}^d \theta_j \in \mathbb{Z}/m$ (it was introduced by E. Zaslow in [17]).
Definition: The *orbifold generalized Euler characteristic* $[X, G]$ of the pair (X, G) is

$$[X, G] := \sum_{c\in G_*} \sum_{\alpha_c=1}^{N_c} [X_{\alpha_c}^g / C_G(g)] \cdot \mathbb{L}^{F_{\alpha_c}^g} \in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{1/m}].$$

Applying the Euler characteristic morphism one gets the notion of orbifold Euler characteristic invented in the study of string theory of orbifolds by L. Dixon et al. [6]:

$$\chi(X,G) := \sum_{c\in G_*}\sum_{\alpha_c=1}^{N_c}\chi(X^g_{\alpha_c}/C_G(g)) = \sum_{c\in G_*}\chi(X^g/C_G(g))\,.$$

Applying the Hodge–Deligne polynomial one gets the orbifold E-function introduced by V. Batyrev in [1]:

$$E_{orb}(X,G;u,v) := \sum_{c\in G_*}\sum_{\alpha_c=1}^{N_c} e(X^g_{\alpha_c}/C_G(g))(u,v)\,(uv)^{F^g_{\alpha_c}} \in \mathbb{Z}[u^{1/m},v^{1/m}]\,.$$

Let $G^n = G\times\ldots\times G$ be the Cartesian power of the group G. The symmetric group S_n acts on G^n by permutation of the factors: $s(g_1,\ldots,g_n) = (g_{s^{-1}(1)},\ldots,g_{s^{-1}(n)})$. The *wreath product* $G_n = G\sim S_n$ is the semidirect product of G^n and S_n defined by the described action. Namely the multiplication in the group G_n is given by the formula $(g,s)(h,t) = (g\cdot s(h), st)$, where $g,\ h\in G^n$, $s,\ t\in S_n$. The group G^n is a normal subgroup of G_n via the identification of $g\in G^n$ with $(g,1)\in G_n$. For a variety X with a G-action, there is the corresponding action of the group G_n on the Cartesian power X^n given by the formula

$$((g_1,\ldots,g_n),s)(x_1,\ldots,x_n) = (g_1x_{s^{-1}(1)},\ldots,g_nx_{s^{-1}(n)})\,,$$

where $x_1,\ldots,x_n\in X$, $g_1,\ldots,g_n\in G$, $s\in S_n$. One can see that the factor variety X^n/G_n is naturally isomorphic to $(X/G)^n/S_n$. In particular, $[X^n/G_n] = [(X/G)^n/S_n]$ in $K_0(\mathcal{V}_{\mathbb{C}})$. Therefore

$$\sum_{n\geq 0}[X^n/G_n]t^n = (1-t)^{-[X/G]} \in K_0(\mathcal{V}_{\mathbb{C}})[[t]]\,.$$

Theorem 4 *(cf. [15],[16]) Let X be a smooth quasi-projective complex algebraic variety of dimension d with an action of a finite group G of order m. Then*

$$\sum_{n\geq 0}[X^n,G_n]t^n = \left(\prod_{r=1}^{\infty}(1-\mathbb{L}^{(r-1)d/2}t^r)\right)^{-[X,G]}\,. \tag{9}$$

Proof. One can say that essentially the proof is already contained in [16] where invariants of the G_n-action on the space X^n are related to those of the G-action on the space X (see also [15] and [14]).

Let $a = (g, s) \in G_n$, $g = (g_1, \ldots, g_n)$. Let $z = (i_1, \ldots, i_r)$ be one of the cycles in the permutation s. The *cycle-product* of the element a corresponding to the cycle z is the product $g_{i_r} g_{i_{r-1}} \ldots g_{i_1} \in G$. The conjugacy class of the cycle-product is well-defined by g and s. For $c \in G_*$ and $r \geq 0$, let $m_r(c)$ be the number of r-cycles in the permutation s whose cycle-products lie in c. Let $\rho(c)$ be the partition which has $m_r(c)$ summands equal to r, and let $\rho = (\rho(c))_{c \in G_*}$ be the corresponding partition-valued function on G_*. One has

$$\|\rho\| := \sum_{c \in G_*} |\rho(c)| = \sum_{c \in G_*, r \geq 1} r m_r(c) = n \, .$$

The function ρ or, equivalently, the data $\{m_r(c)\}_{r,c}$ is called the *type* of the element $a = (g, s) \in G_n$. Two elements of the group G_n are conjugate to each other iff they are of the same type.

In [16] it is shown that:

1. For a conjugacy class of elements of the group G_n containing an element a of type $\rho = \{m_r(c)\}_{r \geq 1, c \in G_*}$ ($\sum_{r,c} r m_r(c) = n$), the subspace $(X^n)^a$ can be naturally identified with $\prod\limits_{c,r} (X^c)^{m_r(c)}$. The factor space $(X^n)^a / Z_{G_n}(a)$ is naturally isomorphic to $\prod\limits_{c \in G_*, r \geq 1} S^{m_r(c)} \left(X^c / Z_G(c) \right)$. The connected components of the space $(X^n)^a / Z_{G_n}(a)$ are numbered by sets of integers $(m_{r,c}(1), \ldots, m_{r,c}(N_c))$ satisfying the relation $\sum\limits_{\alpha_c = 1}^{N_c} m_{r,c}(\alpha_c) = m_r(c)$. They are

$$(X^n)^a_{\{m_{r,c}(\alpha_c)\}} = \prod_{c \in G_*, r \geq 1} \prod_{\alpha_c = 1}^{N_c} S^{m_{r,c}(\alpha_c)} (X^c_{\alpha_c} / Z_G(c)) \, .$$

2. The shift for the component $(X^n)^a_{\{m_{r,c}(\alpha_c)\}}$ is equal to

$$F_{\{m_{r,c}(\alpha_c)\}} = \sum_{c \in G_*, r \geq 1} \sum_{\alpha_c = 1}^{N_c} m_{r,c}(\alpha_c) \left(F^c_{\alpha_c} (r-1) d/2 \right) \, .$$

These two facts imply that

$$\begin{aligned}
\sum_{n\geq 0}[X^n,G_n]t^n &= \sum_{n\geq 0}\left(\sum_{m_r(c)}\prod_{c,r}\prod_{\alpha_c=1}^{N_c}[S^{m_{r,c}}(X^g_{\alpha_c}/Z_G(g))]\mathbb{L}^{m_r(c)(F^g_{\alpha_c}+\frac{(r-1)d}{2})}\right)t^n\\
&= \sum_{m_r(c)}\prod_{c,r}\left(\prod_{\alpha_c=1}^{N_c}[S^{m_{r,c}(\alpha_c)}\left(X^g_{\alpha_c}/Z_G(g)\right)\mathbb{L}^{m_r(c)(F^g_{\alpha_c}+\frac{(r-1)d}{2})}\right)t^{rm_r(c)}\\
&= \prod_{c,r}\prod_{\alpha_c=1}^{N_c}\left(\sum_{m_{r,c}(\alpha_c)}[S^{m_{r,c}(\alpha_c)}\left(X^g_{\alpha_c}/Z_G(g)\right)]\mathbb{L}^{m_r(c)(F^g_{\alpha_c}+\frac{(r-1)d}{2})}t^{rm_{r,c}(\alpha_c)}\right)\\
&= \prod_{c,r}\prod_{\alpha_c=1}^{N_c}\left(1-\mathbb{L}^{(F^g_{\alpha_c}+\frac{(r-1)d}{2})}t^r\right)^{-[X^g_{\alpha_c}/Z_G(g)]}\\
&= \prod_{c,r}\prod_{\alpha_c=1}^{N_c}\left(1-\mathbb{L}^{\frac{(r-1)d}{2})}t^r\right)^{-\mathbb{L}^{F^g_{\alpha_c}}[X^g_{\alpha_c}/Z_G(g)]}\\
&= \prod_{r\geq 1}\left(1-\mathbb{L}^{\frac{(r-1)d}{2})}t^r\right)^{-[X,G]} = \prod_{r\geq 1}\left(1-(\mathbb{L}^{\frac{d}{2}}t)^r\right)^{-\mathbb{L}^{-d/2}[X,G]}.
\end{aligned}$$

□

Taking the Euler characteristic of the both sides of the equation (9), one gets Theorem 5 of [15]:

$$\sum_{n\geq 0}\chi(X^n,G_n)\,t^n = \prod_{r=1}^{\infty}(1-t^r)^{-\chi(X,G)}.$$

Applying the Hodge–Deligne polynomial homomorphism, one gets the main result of [16]:

$$\sum_{n=1}^{\infty}e(X^n,G_n;u,v)\,t^n = \prod_{r=1}^{\infty}\prod_{p,q}\left(\frac{1}{(1-u^pv^qt^r(uv)^{(r-1)d/2})}\right)^{e^{p,q}_{(X,G)}}.$$

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